Emergent Electroweak Gravity

Bob McElrath[1, *]

[1] *CERN theory group, Geneva 23, CH 1211, Switzerland*

We show that any cosmological relic with small self-interactions becomes a superfluid shortly after decoupling, due to the broadening of its wave packet, and lack of any elastic scattering. The dynamics of a superfluid are given by the excitation spectrum of bound state quasi-particles, rather than the center of mass motion of constituent particles. If this relic is a fermion with a repulsive interaction mediated by a heavy boson, such as the Z^0 with neutrinos or dark matter, the condensate has the same quantum numbers as the vierbein of General Relativity. Because there exists an enhanced global symmetry $SO(3,1)_{space} \times SO(3,1)_{spin}$ among the fermion's self-interactions, broken only by it's kinetic term, the long wavelength fluctuations around this condensate is a Goldstone graviton. A gravitational theory exists in the low energy effective theory below the weak scale. These dynamics are an unavoidable consequence of the Standard Model.

INTRODUCTION

In the early universe, relics including photons, neutrinos and dark matter evolve out of thermal equilibrium as their interaction strength becomes small at low temperature in a process known as "freeze-out". This calculation is essentially classical, assuming particles are point-like and using the Boltzmann equation. [6, 14]

After freeze-out the number density of particles is fixed, and the temperature just evolves with Hubble expansion. Their time evolution is given only by the free particle kinetic term. It is usually assumed that the interaction strength is so weak that it can be neglected and that particles remain localized point particles forever. The free particle Hamiltonian propagates the particle and also broadens its wave packet, described by its uncertainty Δx. This is due to the fact that the localization of particles causes them to not be an eigenstate of the Hamiltonian for massive particles.

There are two limits of interest for the particle uncertainty Δx. The free particle approximation is given by $\Delta x \ll n^{-1/3}$. Elastic scattering collisions and the Boltzmann equation describe this system. The opposite limit, $\Delta x \gg n^{-1/3}$ is the quantum liquid approximation. Because particles have wave function overlap with their neighbors, one must take into account collective effects due to contact interactions. If there exists an attractive interaction in any partial wave, then the vacuum energy can be lowered by forming bound state quasi-particles. The system will undergo a phase transition to a superfluid phase, described by the quasi-particles.

If the system contains global symmetries that are broken when the system becomes a superfluid, then goldstone bosons will emerge. As these are massless, their dynamics are extremely important. The global symmetries we will be concerned with are the $SO(3,1)$ coordinate space Lorentz symmetry, and the $SO(3,1)$ spin Lorentz symmetry.

The idea of gravity emerging from spinors is not new and fairly obvious, as one can construct a spin-2 particle as the direct product of spinors [10, 12]. However no workable theory has been yet constructed. The first idea of this type is due to Bjorken [1], who attempted to formulate the photon and graviton as a composite state. The most recent attempt and the most successful is due to Hebecker and Wetterich [7, 16]. Their theory can be regarded as a reformulation of gravity in terms of spinors, but they give no dynamics for the spinors which would lead to such a theory. Furthermore the global symmetry structure of their theory is different from the one presented here. This line of research was largely killed by the paper of Weinberg and Witten [15], which showed that a spin-2 particle could not couple to a covariant conserved current. Two ways out of this theorem are to quantize geometry (the approach of String Theory), or to abandon diffeomorphism invariance as an exact symmetry. Sakharov originally suggested that the graviton could be emergent, and in such theories, diffeomorphism invariance can only be approximate [13].

QUANTUM LIQUID TRANSITION

The quantum liquid regime for a fluid occurs when

$$\Delta x \gg n^{-1/3}. \tag{1}$$

In this limit the system is not classical, and the condition of scattering theory that the impact parameter $b \gg \Delta x$ cannot be satisfied (often known as the "well-localized" assumption).

The expansion of a free particle wave packet in time is

$$\Delta x(t)^2 = \Delta x_0^2 + \Delta v^2 t^2. \tag{2}$$

This can be intuitively understood because different momentum components may move with different velocities. The wave number at $p + \Delta p$ has a velocity $(p + \Delta p)/E$ while the wave number at $p - \Delta p$ has a smaller velocity $(p - \Delta p)/E$ and these two wave numbers may separate in space as they propagate.

The condition for the time-independent superfluid transition can be derived by neglecting the second term of Eq. 2. In the nonrelativistic limit one arrives at

$$T < \frac{\lambda^2 n^{2/3}}{3mk_B}. \tag{3}$$

The cross-section does not enter into this calculation, and the uncertainty Δx_0 is assumed to be proportional to the thermal de Broglie wavelength, $\Delta x_0 = \lambda/p = \lambda/\sqrt{3mkT}$, where λ is an $\mathcal{O}(1)$ parameter reflecting how "localized" the state is. This temperature may be further suppressed by elastic collisions, which must occur frequently enough to keep particles localized to their thermal de Broglie wavelength, but not so often that they destroy the condensate.

In the relativistic case, we also use Eq. 2, however the velocity uncertainty for relativistic states is

$$\Delta v = \frac{\Delta p}{E}(1 - v^2) \tag{4}$$

where $v = p/E$. This correctly reflects the relativistic limit, $v \to c$; massless wave packets do not broaden as each wave number propagates with the same velocity, $v = c$.

The relevant time scale for wave packet broadening is the mean time between collisions $\tau = 1/\sigma n v$ since the uncertainty of a wave packet Δx_0 is set by the 3-momentum of an elastic scattering collision. The condition for a quantum liquid is then

$$\frac{1}{p^2} + \frac{(1-v^2)^2}{\sigma^2 n^2} > \frac{1}{\lambda^2 n^{2/3}}. \tag{5}$$

In the limit that the first term on the left side is small compared to the second (e.g. for decoupled relics with σ small), the condition is independent of temperature:

$$\sigma < \frac{\lambda(1-v^2)}{n^{2/3}}. \tag{6}$$

Thus, for any decoupled cosmological relic, it becomes a quantum liquid when its cross section is approximately less than the square of the inter-particle separation. This occurs faster for non-relativistic relics $v \to 0$ than relativistic ones $v \to 1$, and can be delayed if collisions are "well-localized" relative to the inter-particle separation ($\lambda \to 0$).

This condition (Eq.6) is extremely well satisfied for massive neutrinos and WIMP dark matter, so that today, massive dark matter and at least two neutrino mass eigenstates are definitely quantum liquids.

If attractive contact interactions exist, the system will make a phase transition to a super-fluid in exactly the same way as a BCS superconductor or ^{3}He. For WIMP dark matter, the required contact interaction occurs by integrating out any heavy particles which couple the WIMP to the SM.

Collisions are so rare that they don't break up the collective excitations of the super-fluid, and the relevant condensation criterion is not given by the thermal wavelength (Eq. 3) but rather the time-expanded wave packet as in Eq. 6.

An important implication of this result is that non-relativistic relics such as Dark Matter must be treated as super-fluids. The N-body simulations with point particles propagating in the galaxy, and giving rise to the flattening of galactic rotation curves is a calculation resting on the assumption that dark matter particles are localized, which is incorrect. The time evolution of non-interacting quantum states is important, as are the infrared divergences on the Fermi surface caused by infinitesimal attractive interactions.

THE KOHN-LUTTINGER EFFECT

Beyond wave-function overlap, a necessary condition for a super-fluid state is the existence of a ground state with lower energy than the original vacuum Lagrangian. In the case of an attractive 4-fermion interaction, there obviously exists a lower energy ground state where the fermions bind into s-wave quasi-particles. For Dark Matter theories this is a possibility.

However for the Standard Model, neutrino self-interactions are repulsive.[2] Thus there is no s-wave condensate. However Kohn and Luttinger showed [9] that even a repulsive quantum liquid cannot behave as a classical gas. The reason is that at one loop, 4-point interactions induce a singularity at the Fermi surface that is attractive.[3, 8] Since higher partial wave interactions are exponentially suppressed relative to the s-wave, and this correction scales only as l^{-4}, for some large l this correction dominates. For cosmological relics this can occur already in the p wave.

The relevant correction comes from an exchange (box) diagram and its contribution to the BCS potential $V(x)$ in the lth partial wave is

$$\delta V_l = (-1)^{l+1} \frac{m p_F}{4\pi^2} \frac{|V(\cos\theta = -1)|^2}{l^4} \tag{7}$$

where $p_F = (3\pi n)^{1/3}$ in terms of the number density n. This is attractive for odd l, in terms of the tree-level potential $V(\cos\theta)$ evaluated on the Fermi surface. The relevant divergence occurs for $\cos\theta = -1$ and corresponds to an exchange of the propagating neutrino with a background neutrino. The divergence occurs at $2p_F$ because it occurs in the internal loops, which contain two fermion propagators, both of which must lie on the Fermi surface.

The effective potential in the p-wave for neutrinos on the Fermi surface is then

$$V_1 = \frac{g_Z^4 m p_F}{4\pi^2 M_Z^4}$$

where g_Z is the self-coupling of neutrinos and M_Z is the mass of the Z boson. This is parametrically $\mathcal{O}(p_F^2 G_F^2)$. Therefore this condensation is a much more important effect than scattering, which is associated with the mean free path and is $\mathcal{O}(p_F^5 G_F^2)$. Note that V_1 is also parametrically the same order as Newton's constant G_N, and while firm predictions cannot yet be made because m and p_F are not measured, this is the correct order of magnitude to be the actual G_N.

Therefore, an attractive self-interaction always exists in a neutrino or dark matter fermionic fluid, regardless of the sign of the fundamental interaction. If the mass is sufficiently small so that the conditions of the previous section are also satisfied, then such cosmological relic is a super-fluid today. The two heavier neutrino species and WIMP dark matter, are super-fluids today. Lighter species such as axions and the lightest neutrino (if sufficiently light) would require an early-universe analysis to determine if the conditions of the previous section can be satisfied.

CONDENSATES

The condensates for a fermionic quantum liquid are dictated by Lorentz invariance. A Weyl fermion condenses as $(\frac{1}{2},0)\otimes(\frac{1}{2},0) = (0,0)\oplus(1,0)$ according to their representation under the spin Lorentz group. This gives the one-derivative bilinears

$$A_\mu(x,y) \;=\; \frac{i}{2}(\tilde{\partial}_\mu \chi\epsilon\xi - \chi\epsilon\tilde{\partial}_\mu\xi); \tag{8}$$

$$E^a_\mu(x,y) \;=\; \frac{i}{2}(\tilde{\partial}_\mu \chi^\dagger\overline{\sigma}^a\xi - \chi^\dagger\overline{\sigma}^a\tilde{\partial}_\mu\xi), \tag{9}$$

where $\tilde{\partial}_\mu$ represents the deviation in momentum from the Fermi surface, $p_0 = 0$, $|\vec{p}| = 2p_F$. In condensed matter nomenclature, these excitations are "zero-sound".

The four-point operator for these two condensates is the same since they are related by a Fierz transformation, therefore we may write it as

$$-\frac{g_Z^4 m p_F}{4\pi^2 M_Z^4}\int_{xy}\left[(1-\eta_\nu)E^{a\dagger}_\mu E^\mu_a + \eta_\nu A^\dagger_\mu A^\mu\right]. \tag{10}$$

where

$$\eta_\nu = \frac{n_\nu - n_{\overline{\nu}}}{n_\nu + n_{\overline{\nu}}} \tag{11}$$

is the asymmetry between neutrinos and anti-neutrinos. After the phase transition (Eq.6) has occured, the original Fermi gas is described by momentum distribution functions for A_μ and E^a_μ, rather than original one for free fermions.

The condensate E^a_μ contains both particles and antiparticles, while A_μ contains only particles (or antiparticles). Therefore, A_μ only condenses among the unpaired particles that don't have an antiparticle partner. The Cosmic Neutrino Background (CNB) is expected to contain very nearly equal numbers of neutrinos and anti-neutrinos. The asymmetry η_ν is proportional to the baryon to photon ratio, $\eta_b \sim 6\times 10^{-10}$. Therefore E^a_μ is the dominant condensate and the dynamics of A_μ are strongly sub-leading. A right-handed neutrino state (if they are Dirac) has interactions that are much weaker than the left-handed state, and can be ignored. Likewise, repulsive Majorana dark matter such as a bino is usually not assumed to have any matter/antimatter asymmetry and again can be treated as a single weyl spinor super-fluid which condenses into E^a_μ.

LORENTZ BREAKING

The condensation of A_μ and E^a_μ breaks Poincaré invariance, since both fields have Lorentz indices. This symmetry breaking is dynamical and spontaneous, due to the condensation in a physical background. The fundamental theory is Poincaré invariant. As a consequence of the symmetry breaking, both are Goldstone bosons. A_μ corresponds to a relative gradient in the wave function of two neutrinos. An expectation value for A_μ represents a deviation from constant wave functions, and an expectation value for E^a_μ represents a gradient in the spin density distribution. As such, the expectation value for both is due to the primordial density fluctuations. Long wavelength fluctuations about the expectation values for A_μ and E^a_μ are gapless goldstone bosons.

A free fermion $\psi(x)$ transforms with two Lorentz symmetries. The first is defined on the coordinates of space-time, with the generators

$$L_{\mu\nu} = i(x_\mu\partial_\nu - x_\nu\partial_\mu).$$

Under this symmetry ψ transforms as a scalar. The second Lorentz symmetry is defined with the generators

$$S_{ab} = \frac{i}{2}(\gamma_a\gamma_b - \gamma_b\gamma_a)$$

under which ψ transforms in the 1/2 (spinor) representation. Normally we consider these to be two different representations of the same $SO(3,1)$ Lorentz symmetry. The Lagrangian will not necessarily be symmetric under both groups separately. We will write always Greek indices for the space-time Lorentz group, and Roman indices for the spinor Lorentz group to indicate the difference. Since both groups contain the Minkowski metric $\eta_{\mu\nu}$ and η_{ab}, we will use this to raise and lower indices. The fundamental vacuum theory is Lorentz invariant and has no classical GR background.

A priori, there is no reason these two different groups with different generators should be identified. Since they commute and obey the same algebra, one can therefore define the mixed generators

$$M_{\mu\nu} = L_{\mu\nu} + S_{ab}\delta^a_\mu\delta^b_\nu; \qquad N_{\mu\nu} = L_{\mu\nu} - S_{ab}\delta^a_\mu\delta^b_\nu \quad (12)$$

as is usual in Field Theory. The new operator $N_{\mu\nu}$ is the broken generator, and corresponds for a massless fermion to local violations of being in a helicity eigenstate. A plane wave could be a helicity eigenstate, but a localized state is not an energy or momentum eigenstate, and therefore is also cannot be a helicity eigenstate unless it is completely delocalized. We have written the mixed generators in a suggestive way using δ^a_μ which allows us to tie together indices from the two groups. For this symmetry breaking to occur, there must be an order parameter which gets a vacuum expectation value proportional to δ^a_μ. The indices of δ^a_μ indicate that the field breaking this symmetry must transform as a vector under both symmetries.

Now we must ask what symmetry is obeyed by the effective action. Neutrino self-interactions are mediated by the Z boson. In the Feynman gauge we may write the tree level effective 4-point operator as

$$-\frac{g^2}{2M^2}\int_{xy}\left\{\chi^\dagger\overline{\sigma}^a\chi\xi^\dagger\overline{\sigma}_a\xi\right\}. \quad (13)$$

This interaction has the enhanced symmetry $SO(3,1)\times SO(3,1)$. The only term that breaks this enhanced symmetry is the fermion's kinetic term, which ties together a derivative and a gamma or sigma matrix of the spin Lorentz group.

$$i\int_x \chi^\dagger\overline{\sigma}^\mu\partial_\mu\chi = \int_{xy} E^a_\mu\delta^\mu_a\delta^4(x-y) \quad (14)$$

However this term is a tadpole for the condensate E^a_μ. As such, when E^a_μ condenses, the minimum of the effective action must be shifted $E^a_\mu \to \tilde{E}^a_\mu$ to remove this tadpole, and $\tilde{E}^a_\mu$ is the order parameter of the $SO(3,1)\times SO(3,1)$ symmetry breaking. In the limit that $\tilde{E}^a_\mu \to 0$, the effective action has this enhanced symmetry (and the fermion has no kinetic energy).

By goldstone's theorem, a vacuum expectation value for E^a_μ not only breaks the symmetry $SO(3,1)\times SO(3,1) \to SO(3,1)$, but also generates goldstone bosons from the broken symmetry generators. Here care must be taken because the number of goldstones is not the same as the number of broken generators, because the broken symmetry is a spacetime symmetry. [4, 5, 11]

The goldstones are the long-wavelength fluctuations of the order parameter $\tilde{E}^a_\mu$, and carry a representation of the unbroken group $M_{\mu\nu}$. The field $\tilde{E}^a_\mu$ however carries an index of both the original groups. The propagating goldstone is

$$g_{\mu\nu} = \tilde{E}^a_\mu\tilde{E}^b_\nu\eta_{ab} \quad (15)$$

which we identify as the graviton. This should be familiar from the Palatini formalism for quantizing gravity, if we identify $\tilde{E}^a_\mu$ as the vierbein (tetrad).

The gravitational theory arising here does not conflict with the Weinberg-Witten Theorem [15] because of the presence of a physical background, and consequently this theory isn't diffeomorphism invariant. While a spin-2 field cannot couple to a conserved current in flat Minkowski space, the presence of the background, and the fact that E^a_μ is a fluctuation in that background mean that this graviton does not live in Minkowski space. It only lives in the curved space defined by its density and spin distribution.

From here one can almost directly follow the program of "Spinor Gravity" [7, 16], with the exception that we consider matter transforming under $SO(3,1)$ with $L_{\mu\nu}$, and therefore we have the metric $\eta_{\mu\nu}$ with which to tie up spacetime indices, where the program of Spinor Gravity uses $GL(4)$ instead of $SO(3,1)$, as a consequence they do not have a spin connection, where we can define the spin connection with $\tilde{g}^{ab} = E^a_\mu E^b_\nu\eta^{\mu\nu}$ in the usual way

$$\omega^{ab}_\mu = E^a_\nu\partial_\mu E^{\nu b} + E^a_\nu E^{\sigma b}\Gamma^\nu_{\sigma\mu}. \quad (16)$$

The existence of $\eta_{\mu\nu}$ implies more invariants as well. However due to the global rather than local Lorentz invariance of the spinor index, the present theory has torsion.

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* `bob.mcelrath@cern.ch`

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